Indian Statistical Institute,Bangalore Centre
B.Math.(Hons.)II Year - 2016-17, First Semester
Optimization

Back Paper Exam — 30 December 2016, 10 am -1 pm
Instructor: P.S.Datti — Max.Marks: 100

NOTE:

(i) Answer *any* 7 in Questions $1-9$ and answer Question 10. WRITE NEATLY.

(ii) $M_n(\mathbb{R})$ denotes the set of real $n \times n$ matrices.

1. Let $\mathbf{e}_j$, $1 \leq j \leq n$, denote the standard unit vectors in $\mathbb{R}^n$. Put $\mathbf{E}_{ij} = \mathbf{e}_i\mathbf{e}_j^t$ for $1 \leq i, j \leq n$. Define
$$\mathbf{A} = \sum_{i=1}^{n-1} \alpha_i \mathbf{E}_{i,i+1},$$
where α_i for $i = 1, \ldots, n-1$ are real numbers. Show that $\mathbf{I} - \mathbf{A}$ is invertible and obtain an expression for $(\mathbf{I} - \mathbf{A})^{-1}$ in terms of powers of $\mathbf{A}$. **(12)**

2. (a) Find all the 2×2 projection matrices. **(4)**

   (b) Suppose $\mathbf{A}$ is a real $m \times n$ matrix of rank n and define $\mathbf{P} = \mathbf{A}(\mathbf{A}^t\mathbf{A})^{-1}\mathbf{A}^t$.

   i. Verify that the definition of $\mathbf{P}$ makes sense and show that $\text{im}\mathbf{P} = \text{im}\mathbf{A}$ and $\ker \mathbf{P} = \ker \mathbf{A}^t$. **(1+4)**

   ii. Show that $\mathbf{P}$ is an orthogonal projection. **(3)**

3. (a) Let $\{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_k\}$ be a set of orthonormal (column) vectors in $\mathbb{R}^n$ and $\mathbf{x} \in \mathbb{R}^n$. Show that
$$\|\mathbf{x} - \sum_{i=1}^{k} \lambda_i \mathbf{u}_i\| \geq \|\mathbf{x} - \sum_{i=1}^{k} (\mathbf{x}, \mathbf{u}_i)\mathbf{u}_i\|,$$
for all real numbers $\lambda_1, \ldots, \lambda_k$, with equality holding if and only if $\lambda_i = (\mathbf{x}, \mathbf{u}_i)$ for $i = 1, \ldots, k$. **(6)**

   (b) Let $\mathbf{A}$ be an $m \times n$ real matrix and $\mathbf{b}$ is a given column vectors in $\mathbb{R}^m$. If $\mathbf{P}$ is the orthogonal projection onto $\text{im}\mathbf{A}$, show that
$$\|\mathbf{A}\mathbf{x} - \mathbf{b}\| \geq \|\mathbf{b} - \mathbf{P}\mathbf{b}\|,$$
for all $\mathbf{x} \in \mathbb{R}^n$, with equality holding if and only if $\mathbf{A}\mathbf{x} = \mathbf{P}\mathbf{b}$ for some $\mathbf{x} \in \mathbb{R}^n$. **(6)**

4. (a) Let $\mathbf{A} = ((a_{ij})) \in M_n(\mathbb{R})$ be a positive matrix, that is $a_{ij} > 0$ for all i, j. Suppose there is a $\lambda > 0$ and a vector $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$ such that $\mathbf{A}\mathbf{x} > \lambda\mathbf{x}$. Show that there is a $\delta > 0$ such that $\mathbf{A}\mathbf{x} \geq (\lambda + \delta)\mathbf{x}$. **(4)**

(b) Suppose $\mathbf{A} \in M_n(\mathbb{R})$. Show that $\mathbf{A}$ is a positive matrix if and only if $\mathbf{Ax} > \mathbf{0}$ for all $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$. **(4)**

(c) Suppose $\mathbf{A} = ((a_{ij})) \in M_n(\mathbb{R})$ is a non-negative matrix, that is $a_{ij} \geq 0$ for all i, j and $\mathbf{x} \geq \mathbf{0}$, $\mathbf{x} \neq \mathbf{0}$. Define

$$\begin{aligned} r_{\mathbf{x}} &= \min_{1 \leq i \leq n} \left\{ \frac{(\mathbf{Ax})_i}{(\mathbf{x})_i} : (\mathbf{x})_i > 0 \right\}, \\ R_{\mathbf{x}} &= \max \left\{ \lambda \geq 0 : \mathbf{Ax} \geq \lambda \mathbf{x} \right\}. \end{aligned}$$

Show that $r_{\mathbf{x}} = R_{\mathbf{x}}$. **(4)**

5. (a) Show that 1 is the dominant eigenvalue of the doubly stochastic matrix $\mathbf{A} = \begin{pmatrix} 1/2 & 0 & 1/3 & 1/6 \\ 1/6 & 1/2 & 1/3 & 0 \\ 0 & 1/6 & 1/2 & 1/3 \\ 1/3 & 0 & 1/6 & 1/2 \end{pmatrix}$ and hence find the limit $\lim_{k \to \infty} \mathbf{A}^k$. **(6)**

(b) Suppose $\mathbf{A} \in M_n(\mathbb{R})$ is a non-negative, irreducible matrix such that its spectral radius equals 1. Prove the following:

i. There are positive vectors $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ such that $\mathbf{Au} = \mathbf{u}$ and $\mathbf{A}^t \mathbf{v} = \mathbf{v}$, with $\mathbf{v}^t \mathbf{u} = 1$. **(3)**

ii. Define $\mathbf{B} = \mathbf{A} - \mathbf{uv}^t$ and show that each non-zero eigenvalue of $\mathbf{B}$ is also an eigenvalue of $\mathbf{A}$, but 1 is *not* an eigenvalue of $\mathbf{B}$. **(3)**

6. (a) Solve the following using simplex method:

$$\begin{aligned} &\text{maximize } 3x_1 + x_2 + 3x_3 \\ &\text{subject to } 2x_1 + x_2 + x_3 \leq 2 \\ &\qquad x_1 + 2x_2 + 3x_3 \leq 5 \\ &\qquad 2x_1 + 2x_2 + x_3 \leq 6 \\ &\qquad\quad x_i \geq 0,\ i = 1, 2, 3. \end{aligned}$$

(6)

(b) Let $\mathbf{A}$ be an $m \times n$ real matrix and $\mathbf{b}, \mathbf{c}$ are given column vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Consider the following linear programming:

$$\text{minimize } \mathbf{c}^t \mathbf{x}, \text{ subject to } \ \mathbf{Ax} = \mathbf{b},\ \mathbf{x} \geq \mathbf{a},$$

where $\mathbf{a} \geq \mathbf{0}$ is a given vector in $\mathbb{R}^n$. Find the dual problem. **(6)**

7. (a) Suppose C is a convex set in $\mathbb{R}^n$ and $k \geq 2$. Let $\mathbf{x}_1, \ldots, \mathbf{x}_k$ are in C and $t_1, \ldots, t_k$ are non-negative real numbers such that $t_1 + \cdots + t_k = 1$. Show that $t_1 \mathbf{x}_1 + \cdots + t_k \mathbf{x}_k$ is also in C. **(3)**

(b) State the Farkas-Minkowski lemma and prove it using the duality theorem of the linear programming.. **(6)**

(c) Using the theorem of the alternative, show that the following system

$$\begin{pmatrix} 1 & 3 & -5 \\ 1 & -4 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 3 \end{pmatrix}$$

does not have a non-negative solution. **(3)**

8. Consider the following LPP:

$$\begin{aligned} \text{minimize} \ & -x_1 - 4x_2 - 3x_3 \\ \text{subject to} \ & 2x_1 + 2x_2 + x_3 \leq 4 \\ & x_1 + 2x_2 + 2x_3 \leq 6 \\ & x_i \geq 0, \ i = 1, 2, 3. \end{aligned}$$

It is given that an optimal solution of the above is $(0, 1, 2)$.

(a) Write down the corresponding dual problem. **(4)**

(b) Solve the dual problem by the simplex method. **(6)**

(c) Verify the duality theorem for these duality relations. **(2)**

9. Let $\mathbf{A}$ be a real $m \times n$ matrix. Show that the system $\mathbf{Ax} > \mathbf{0}$ has a solution if and only if the following statement holds true:

$$\boldsymbol{\lambda} \in \mathbb{R}^m, \ \boldsymbol{\lambda} \geq \mathbf{0} \text{ and } \boldsymbol{\lambda}^t\mathbf{A} = \mathbf{0} \Rightarrow \boldsymbol{\lambda} = \mathbf{0}.$$

(12)

10. (a) Let $\mathbf{A}$ be an $m \times n$ real matrix and $\mathbf{c}$ is a given column vector $\mathbb{R}^n$. Assume that the following statement holds true:

$$\mathbf{x} \in \mathbb{R}^n, \ \mathbf{Ax} \leq \mathbf{0} \Rightarrow \mathbf{c}^t\mathbf{x} \leq 0.$$

Show that there is a $\boldsymbol{\lambda} \in \mathbb{R}^m$, $\boldsymbol{\lambda} \geq \mathbf{0}$ such that $\mathbf{c}^t = \boldsymbol{\lambda}^t\mathbf{A}$. **(8)**

(b) Let $\mathbf{A}$ be an $m \times n$ real matrix and $\mathbf{b}, \mathbf{c}$ are given column vectors in $\mathbb{R}^m$ and $\mathbb{R}^n$ respectively. Put $k = m + n$ and define the $k \times k$ matrix $\mathbf{M}$ by $\mathbf{M} = \begin{pmatrix} \mathbf{O}_{n\times n} & -\mathbf{A}^t \\ \mathbf{A} & \mathbf{O}_{m\times m} \end{pmatrix}$ and $\mathbf{q} = \begin{pmatrix} \mathbf{c} \\ -\mathbf{b} \end{pmatrix}$. Consider the linear programme (P) given by

$$\begin{aligned} \text{minimize} \ & \mathbf{q}^t\mathbf{z} \\ \text{subject to} \ & \mathbf{Mz} \geq -\mathbf{q} \\ & \mathbf{z} \geq \mathbf{0}. \end{aligned}$$

Show that the problem (P) and its dual are the same. Further, show that any feasible solution of (P) is also optimal. **(8)**